GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN

Research Training Group 2191
Fourier Analysis and Spectral Theory
Research programme and thesis projects

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Contents

1 Overview

Our research programme focuses on modern Fourier analysis and spectral theory that come up in a variety of contexts and run like a common thread through our research projects. As described earlier, our approach to harmonic analysis and spectral theory is interdisciplinary with analytic, topological and arithmetic features, but the underlying objects (such as Riemannian manifolds), ideas (such as Fourier duality) and methods (such as spectral decomposition, microlocal analysis) are common to the entire programme.

We start with a short synopsis of the research projects, along with their interrelations, and put them into perspective. Along the way we will, in particular, demonstrate stable edges between the vertices of the triangle analysis – analytic number theory – topology. Complete details will be given in the forthcoming subsections.

Project areas 2.1 and 2.8 are purely analytic in nature and thus provide a classic focal point for the RTG. The former uses microlocal methods to understand and construct the asymptotic series of perturbative quantum field theory, while pseudodifferential methods for boundary value problems and Fourier integral operators deal with QFT in the presence of boundaries. 2.1 is of an interdisciplinary nature and —while firmly embedded in mathematics— provides the RTG with an important connection to quantum physics which is relevant also for students working, for instance, in project areas 2.5, 2.7 or 2.8. The focus of 2.8 is the resolvent of elliptic (or hypoelliptic) differential operators which encodes important information on their spectral and scattering theory. The underlying analytic methods are fundamental to the entire RTG. In particular, the spectral theory of the Laplacian on nilpotent groups is the analytic heart of the index theory on such spaces studied in 2.6 and for the representation theoretic applications in 2.5. Spectral theory on symmetric spaces is the analytic heart for L^2-invariants of these, which are a topic of 2.6. The dynamical aspects in 2.8 have the potential of arithmetic applications, in the theory of automorphic forms.

Fourier analysis in arithmetic situations comes up in project areas 2.2 and 2.3. The power of Fourier analytic techniques in diophantine analysis is demonstrated in 2.2, featuring modern variants of the Hardy–Littlewood circle method. More combinatorial aspects of Fourier analysis appear, for instance, in sieve theory (see 2.3). The most well-known example is the large sieve, which is nothing but an ℓ^2-operator norm. The project areas 2.2 and 2.3 are linked by the fact that Fourier analysis is enhanced by number theory in the guise of lattice point problems, multiplicative structures (in particular prime numbers) and diophantine considerations which come up in all of them. Put differently, Fourier and harmonic analysis are tailored to encode arithmetic phenomena.

2.2 and 2.3 apply spectral theory in discrete situations. This is also the core of project area 2.4, which employs the Cayley graph Laplacian to study Kazhdan's property (T) for groups. Methodologically, this is linked to optimization and transformation techniques in analytic number theory mentioned in the previous paragraph, but it also builds a bridge to topological questions. This project is one of the special features of our unique group of PIs, building a bridge between number theorist Harald Helfgott and topologist Thomas Schick on the basis of spectral theory and supported by previous work of both PIs. It also profits from the additional expertise of group theorist Laurent Bartholdi as associated researcher. The link to topological questions is given by the fact that the Cayley graph Laplacian is a special case of the combinatorial Laplacian of some cellular L^2-chain complex which comes with corresponding L^2-invariants. In the context of symmetric spaces for semisimple and nilpotent Lie groups, the investigation of these L^2-invariants with tools from harmonic analysis and index theory for certain invariant differential operators is at the core of Project 2.6.

Implicit in many of the projects just discussed – such as 2.4, 2.6 – is the question of the connection between geometric properties of spaces acted on by groups and spectral properties of certain invariant operators (such as the Laplacian). This theme comes up most directly and prominently in 2.7, where we ask to what extent we can choose a metric to achieve a determined band-gap structure of the spectrum of this operator. A key tool here is Fourier analysis in the form of Bloch–Floquet theory.

Project area 2.5 studies Fourier analysis in the more abstract form of representation theory; it centres on variations of Nelson's theorem, describing which representations of certain *-algebras of operators on a manifold extend to the underlying C^*-closure, generalizing the problem of integrating Lie algebra representations to the underlying Lie group. Methodological connections to 2.6 and 2.8 such as pseudodifferential calculi are put in more detailed context in the project description below.

This is a challenging programme that is based on a broad and advanced mathematical machinery. It therefore benefits from an *additional postdoctoral researcher* who can contribute his or her own ideas and specific competences to the RTG and serves as an early career (but more advanced) partner of the doctoral researchers.

We now turn to the detailed and explicit description of the research programme. The following subsections, ordered alphabetically with respect to the first PIs name, describe specific project areas and list a selection of 26 possible thesis projects which provide ample material for two cohorts of 10 PhD students each.

2 Details of the Research programme

2.1 Microlocal methods in quantum field theory (Bahns, Schrohe, Witt)

As of today, no interacting quantum field theory has been constructed in four space-time dimensions. Exact models exist only in two dimensions, and otherwise one resorts to perturbation theory. This yields an asymptotic series composed of terms involving convolutions and products of certain fundamental solutions of the underlying "free theory" (a linear partial differential operator, typically normally hyperbolic, such as the wave operator $\partial_t^2 - \Delta_x$). The terms in this series as they stand are in general ill-defined, and physicists have developed elaborate tools to "renormalise" them, that is, to assign finite values to them.

Most of these tools work only in flat Minkowski space and cannot be generalized to cosmological space-times. This is because they often rely on a globally defined Fourier transform and the existence of a distinguished "vacuum state", both of which do not exist in the generic situation. Moreover, these tools are almost exclusively developed for the case where the underlying partial differential operator is elliptic. This elliptic theory then has to be mapped to the physically relevant hyperbolic situation by a so-called Wick rotation. While the Osterwalder–Schrader positivity property justifies this in some situations, in many cases the Wick rotation cannot be globally defined.

Based on Radzikowski's work [92], Brunetti and Fredenhagen [21] formulated the renormalisation problem in quantum field theory (QFT) in the context of microlocal analysis as a problem of extending (certain) distributions. In this approach, we are given a submanifold Σ of a globally hyperbolic manifold M and a distribution $u \in \mathcal{D}'(M \setminus \Sigma)$ that is conormal with respect to Σ, and we must extend it to a distribution on M while controlling physically relevant parameters such as the scaling degree (a generalised homogeneity degree). Together with the axiomatic approach of Hollands and Wald to perturbation theory on curved space-times, this reformulation paved the way to studying QFT systematically and rigorously in more interesting geometries than Minkowski space. The underlying iterative construction of the perturbative series was further formalized and led to the framework of perturbative algebraic quantum field theory.

One main tool in the microlocal approach is Hörmander's wavefront set, which gives a finer ("microlocal") resolution of the singular support of a distribution. It is a subset of the cotangent bundle, which captures not only the singularities of a distribution, but also the co-directions of high frequency that cause them. It is calculated using a localized Fourier transform. The application of the wavefront set in QFT has since been further developed, among others, in [26, 60] or in Wrochna's Göttingen thesis (compare [DB4]), and it has been applied also to QFT models on the non-commutative Moyal space (see [DB2, DB1]).

Another key ingredient of this approach is a certain class of states on a suitable algebra of functionals, the so-called "Hadamard states" that were first characterized in [92] and replace the vacuum state of flat space-time. Starting from such Hadamard states, quantum fields can be realized as unbounded operators on a Hilbert space and correlation functions can be calculated. Distinguished parametrices and Hadamard states for non-flat Lorentzian space-times were constructed, among others, by Dappiaggi et al [27] and, more recently, by Gérard and Wrochna [40] and Vasy [115].

In recent work, the extension problem was reformulated by Brouder, Dang [26], and others to take into consideration larger and larger classes of distributions. We propose to pursue a different avenue which is to consider the renormalisation problem of QFT in the framework of Lagrangian distributions, that is, distributions which are conormal with respect to a Lagrangian submanifold Λ of the cotangent bundle T^*M. Locally, a Lagrangian distribution is given as an oscillatory integral (ubiquitous also in the analytic theory of automorphic forms),

$$u(x) = \int a(x,\xi)\, e^{i\phi(x,\xi)}\, d\xi$$

with a nondegenerate phase function ϕ whose manifold of stationary phase is contained in Λ. However, the challenge of the global theory (for instance, symbolic calculus) is that it has to be set up in a geometric, coordinate independent way. Lagrangian distributions provide the natural framework for the renormalisation problem as the (distinguished) fundamental solutions of the partial differential operators in question are one-sided paired Lagrangian distributions [35, 77]. For such distributions, a symbolic calculus is available [55]. The symbolic calculus and the microlocal machinery should be combined to study the problem of extending such distributions using the methods of [78].

A thesis project, supervised by Bahns and Witt, then consists in iterating this construction as stipulated by the asymptotic series of perturbative QFT, while keeping track of the structure of the distributions that arise in each step of the iteration, including the symbolic information we have on them. In some sense, this resembles higher-order microlocalisation (see [63]). The details of our iterative procedure involve the construction of specific classes of distributions on certain stratified spaces, and the construction of a functorial extension map defined on such classes of distributions.

Another project, supervised by Bahns, Schrohe and Witt, will concern QFT in the presence of boundaries. There is previous work such as [70] and [23], but a microlocal approach has not yet been fully developed. The thesis project will consider especially the Neumann boundary conditions, where the uniform Lopatinski condition is violated, and it will include the construction of Hadamard states. Here, Schrohe's expertise in pseudodifferential methods for boundary value problems and Fourier integral operators [ES2], [ES6] plays an essential role. He has moreover adressed QFT questions earlier with Junker [ES3] and constructed Hadamard states and adiabatic states on globally hyperbolic space-time manifolds with a compact Cauchy surface in terms of the Sobolev wavefront set.

Regarding constructive aspects in QFT, a result by Bahns and Rejzner [DB3] shows that in the framework of perturbative algebraic quantum field theory, the S-matrix of the Sine Gordon model on 2-dimensional Minkowski space (hyperbolic signature) is constructible as a unitary operator. More recently, Bahns, Fredenhagen and Rejzner have shown that the Haag–Kastler net of von Neumann algebras of local observables can be constructed explicitly [DB5] – and hence, the framework indeed provides a completely new approach to constructive QFT. Until now, results on exact models had mostly been restricted to the elliptic signature case, and a subsequent Wick rotation was needed. One suggested thesis project in this framework, supervised by Bahns, is the construction of the conserved currents of the model. Again, this construction will require renormalisation.

Preliminary titles of thesis projects:

- Renormalisation in terms of paired Lagrangian distributions.
- Microlocal methods for QFT in the presence of boundaries.
- Conserved currents in the Sine Gordon model on Minkowski space.

2.2 Arithmetic Fourier analysis (Brüdern)

Modern diophantine analysis derives its power from a complex mix of tools from harmonic analysis, combinatorics, and aspects of Banach space theory, all of this in rather concrete form. The main question in the area is whether a given family of varieties, affine or projective, but of large dimension, obeys a local-to-global (Hasse) principle. One way to attack this is through Fourier analysis, via the circle method of Hardy and Littlewood or variants thereof (see, for instance, [JB1]). The pivotal contribution by Davenport and Birch [10, 29] remained unimproved over half a century, except in the special case of cubic forms [51, 54]. Very recently, a flurry of ideas emerged, culminating with Meyrson's dramatic work on a large class of complete intersections in the projective world. It is clear that the new machinery is far from running out of steam. Here is one problem, mainly of analytical character, that combines analytic number theory and Fourier analysis:

Given a polynomial $F \in \mathbb{Z}[x_1, \ldots, x_s]$ of degree d, and a solution $x \in \mathbb{Z}^s$ of $F(x) = 0$, estimate the smallest solution of $F(x) = 0$ in terms of d, s and the height of F.

This is surprisingly hard. Leaving aside the trivial case of linear polynomials, a complete solution is only available for $d = 2$ (Dietmann [30]). For cubic forms, one also has a positive answer when $s \geq 17$ [20]. One would expect that when F is a form of degree d and $s > 2^d d$ or so, then again one should be able to establish a bound for the smallest zero of F that is polynomial in the height of F. There is, however, a serious obstacle. For the circle method to succeed, one has to estimate the so-called singular integral from below, in terms of the coefficients of F. While it is straightforward to show that the singular integral is positive, quantitative estimates have not been found. One possible line of attack is to explore a well-known interpretation of the singular integral as a weighted measure for the area of the real surface $F(t) = 0$ with $|t_j| \leq 1$ for $1 \leq j \leq s$. Through this link and an analysis of the geometry of the surface $F(t) = 0$ we expect the following results to be within reach:

1. establish a polynomial estimate for the smallest integer solution of $F(x_1, \ldots, x_s) = 0$ in terms of the height, for forms of degree d in $s > 2^d d$ variables.

2. establish a similar result for cubic polynomials, not necessarily homogeneous, at least when s is as large as about 17.

Another class of problems that is largely of analytic nature derives from joint work of Brüdern and Wooley. The authors of [JB4, JB5] referred to their strategy as arithmetic harmonic analysis. The idea behind this is best illustrated with a simple example. Until recently, a successful use of the circle method depended, in one way or another, on duality principles and Parseval's relation. In some cases, however, one arrives at different moments of Fourier coefficients in a natural way. Suppose we are given three polynomials $F_j(x_1, \ldots, x_s)$ with integer coefficients, and are interested in solutions of $F_1(x) = F_2(y) = F_3(z)$, for simplicity with the coordinates in a large box, say $|x| \leq P$, $|y| \leq P$, $|z| \leq P$. Then one considers the Fourier series

$$S_j(\alpha) = \sum_{|x| \leq P} \mathrm{e}^{2\pi \mathrm{i} \alpha F_j(x)} = \sum_n c_j(n) \mathrm{e}^{2\pi \mathrm{i} \alpha n},$$

where $c_j(n)$ counts solutions of $F_j(x) = n$. Now we have two expressions for the number of solutions of $F_1 = F_2 = F_3$, namely, the sum

$$\sum_n c_1(n) c_2(n) c_3(n)$$

and the dual expression

$$\int_0^1 \int_0^1 S_1(\alpha) S_2(\beta - \alpha) S_3(-\beta) \, \mathrm{d}\alpha \, \mathrm{d}\beta.$$

For an upper bound, one may use Hölder's inequality to get

$$\Big| \sum_n c_1(n) c_2(n) c_3(n) \Big|^3 \leq \prod_{j=1}^{3} \sum_n |c_j(n)|^3.$$

The advantage is that each of the factors on the right depends only on one of the polynomials F_j. So this strategy disentangles the effects of the input polynomials. The disadvantage is that the cubic moment has no immediate diophantine interpretation. This is different from the mean square: here

$$\sum_n |c_j(n)|^2$$

counts certain solutions of the equation $F_j(x) = F_j(y)$. Parseval's identity deals with the equation $F_1(x) = F_2(y)$ in this style.

Of course, even for cubic moments, one could try to work with general results on Fourier coefficients, like the Hausdorff–Young inequalities. These seem to go in the wrong direction, however, or produce trivial estimates only. Hence, for a successful implementation of such ideas, one has to train the Fourier analysis to remember the arithmetic origin of the Fourier coefficients. This was done, for the first time, in a very special case to establish the key lemma in [JB3] (see also [JB2]) and then in other contexts in [JB4, JB5]. However, so far, we only have just a few examples where the method performs, and we are far from a systematic theory. But even then, the ideas behind our analysis are far from being exhausted. The following project is a starting point for a beginner in the area:

Begin with correlation estimates between an exponential sum "of arithmetic origin" and an exponential sum over a polynomial. Equipped with these, systematically find examples where ideas related to [JB4] *lead to improvements over a more routine application of the circle method.*

It is very interesting to combine arithmetic harmonic analysis with the emerging field of additive combinatorics, also sometimes referred to as higher degree Fourier analysis. It may well be that the first can be developed to become a tool for the latter, but at this stage such links can, at best, be explored at an experimental level. While additive combinatorics has produced celebrated results like the Green–Tao Theorem, the more spectacular applications to number theory are mostly limited to the solutions of linear systems with variables from "structured sets," like the primes, or integer sequences that contain no three elements in arithmetic progressions. A brave student should try to take these ideas further, and study some classes of higher degree equations that are favourable to the methods underpinning the Green–Tao Theorem. A good background in Banach space techniques will be required here, on top of a course on the circle method.

Possible thesis problems include

- Small solutions of diophantine equations.
- Diophantine correlation estimates.
- Roth estimates for higher degree equations.

2.3 Primality and parity (Helfgott)

One of the central issues in analytic number theory is the difficulty in distinguishing primes and almost-primes. The *parity problem* asks to distinguish numbers with an even or odd number of prime factors. This problem is important because it encapsulates what we cannot do, or find very hard to do, in our study of the primes. Our techniques are much better at telling apart primes and numbers with many prime factors, or numbers whose number of prime factors differs by considerably more than 1.

Strong recent results on primes have allowed us to gain some ground against the problem. The main two examples are the work of Goldston, Pintz, Yildirim, Zhang and Maynard on gaps between primes (see [73]), and the work of Matomäki, Radziwiłł and Tao [71] on sign changes of the Möbius function. Work in progress by PI Helfgott and Radziwiłł combines spectral analysis with new techniques to go beyond the results in [71].

The proof of the ternary Goldbach problem by PI Helfgott [HH4, HH3] shows that every odd number greater than 5 can be expressed as the sum of three primes. Helfgott's work on a second version of the proof has led to several interesting problems that are suitable for doctoral students. These are problems that lie clearly in analysis, and seem likely to require techniques from harmonic analysis.

The main idea is as follows. Consider one of the main and simplest uses of sieves, namely to single out primes. An upper-bound sieve consists of coefficients λ_d carefully chosen so that

$$\sum_{n\leq x}\left(\sum_{d|n}\mu(d)\lambda_d\right)^2 \tag{1}$$

(with $\mu(n)$ the Möbius function) is as small as possible, that is, not much larger than $\pi(x) \sim x/\log x$, the number of primes up to x.

Given the common constraints $\lambda_1 = 1$, $\lambda_d = 0$ for $d > D$ for a parameter D, it is clear that the expression (1) is at least $\pi(x) - \pi(D)$, since the inner sum in (1) consists only of the term $\lambda_1 = 1$ whenever n is a prime between D and x, and the square of the inner sum is, of course, always non-negative, being a square.

If $D \le \sqrt{x}$, it is not possible to make (1) smaller than about $2\pi(x)$, or rather $x/\log D$, or very slightly less. This is one manifestation of the *parity problem* mentioned above. The optimal choice of weights λ_d was found by Selberg; it depends both on the size of d and its divisibility properties.

Now, what if we impose the constraint that λ_d be the restriction to $\mathbb{Z}$ of a continuous, monotone function on $\mathbb{R}^+$? (Such a constraint is natural; it is imposed to us, for instance, when λ_d arises from a smoothing function used for other purposes, as it does in the study of the ternary Goldbach problem.)

The optimal choice is then not known, though a function studied by Barban and Vehov [4] is a likely candidate. Barban and Vehov showed in 1968 that their function gives a result within a constant factor of the theoretical optimum. Graham showed some ten years later that the constant was asymptotically one. The convergence to the asymptotic could, as far as anybody was aware, be rather slow, due to a relatively poor bound on the second-order term.

In the current version of [HH4], PI Helfgott manages to analyse the case – also studied by Barban–Vehov and Graham – where D is larger than $\sqrt{x}$. This case is out of reach for most sieves – including Selberg's optimal quadratic sieve – but not for this one. It is shown that the second-order term is, in fact, *negative*, and precise bounds for it are proven, with carefully determined explicit constants. Helfgott's doctoral student S. Zúñiga Alterman is currently investigating similar results for the case $D \le \sqrt{x}$. This has plenty of potential applications that will lead to interesting thesis projects.

It is an open question to what extent the choice in [4] is optimal. This problem would be a very good fit for a doctoral student familiar with both Fourier analysis and optimization problems. A very concrete optimization problem also has to be solved in project area 2.4.

Let us make clear why harmonic analysis is a necessary part of the repertoire of someone attacking this problem (or indeed of any analytic number theorist). Fourier analysis is a very common technique in number theory; indeed it forms the backbone of the circle method, used to treat the ternary Goldbach problem since Hardy, Littlewood and Vinogradov. While the circle method would not be the approach to follow for the optimization problem just discussed, Fourier analysis is likely to be useful in other ways. For instance, applying the Poisson summation formula is a completely standard step in this sort of problem. The issue is really when to apply it, and what to do thereafter. The point is to unblock a problem by working in Fourier space, instead of solely in physical space.

The following are possible titles of thesis projects in this direction:

- Optimality among monotonic upper-bound sieves.
- On explicit minor-arc estimates in Goldbach's problem.

2.4 Spectral analysis in geometric group theory (Helfgott, Schick)

Discrete groups are ubiquitous in mathematics as the classical vehicle encoding symmetry. As such, they interact with essentially any other area of mathematics. Vice versa, this leads to a huge arsenal of different tools for their study.

Spectral theory is surprisingly powerful. Here we mean the spectral theory of the Cayley graph Laplacian (for simplicity, we assume that the group is finitely generated), which is equivalent to the spectral theory of the symmetric random walk on the graph. If the Cayley graph is constructed using the generating set S with $|S| = d$, the Laplacian is $\Delta = 2d - \sum_{s\in S}(s + s^{-1})$, acting on ℓ^2-functions on the set Γ of vertices of the Cayley graph. Here group element $\gamma \in \Gamma$ acts by left multiplication: $\gamma \cdot (\sum_{g\in\Gamma} c_g g) = \sum_{g\in\Gamma} c_g(\gamma g)$. Commonly used is the Markov operator $M = 1 - \Delta/(2d)$. Its spectrum captures a lot of information, which is perhaps best encoded in its Green function (a generating power series). For a tree it can be explicitly calculated and turns out to be a rational function. The spectral radius of the random walk, defined as the spectral radius of M, is the radius of convergence of the Green function. So it is given by its pole structure. The technique of analysing singularities of generating functions is also very common in analytic number theory, see 2.2.

The specific group theoretic spectral property we address is Kazhdan's property (T). In our language, property (T) for a group Γ is equivalent to the fact that there is a gap near 1 in the spectrum of M, this time considered as an operator on the direct sum of all irreducible unitary representations of Γ (instead of only the regular representation as before). Equivalently, there is a gap near 1 of the image of M in the maximal group C^*-algebra of Γ. It is clear that this property holds if $\Delta^2 - \epsilon\Delta = \sum_{j=1}^{n} a_j^* a_j \in \mathbb{R}[\Gamma]$ for some $\epsilon > 0$. Surprisingly, Ozawa [83] shows that this algebraic criterion is also necessary for property (T). This has incited recent work to find groups with property (T) (and good Kazhdan constants ϵ) with this method, for instance, for $\mathrm{SL}_3(\mathbb{Z})$ [80]. The first new group where property (T) could be established with this method is $\mathrm{Aut}(F_5)$ [56]. The main point is that to find the positive square representation can be transformed into a problem of semidefinite optimization. This is a quadratic optimization problem, and similar problems are relevant in project area 2.3.

The method has not yet been successful for $\mathrm{Aut}(F_4)$. The result of [56] also reproves and sheds new light on the fact that the finite symmetric groups can be equipped with generators making them a uniform sequence of expanders, first established in [59], related to the work of Helfgott on bounded generation in finite groups [HH1, HH2, HH5, HH6].

We are now interested in the case $\mathrm{Out}(F_4)$. Note that the previous work is purely algebraic. We propose to combine Ozawa's method with geometry. In particular, observe that $\mathrm{Out}(F_n)$ has an explicit model for its universal space for proper actions, outer space, a simplicial complex whose points are metric trees. The action is not cocompact, but there is an explicit subcomplex, the spine, which is universal and cocompact. This leads both to a general question, and to a specific application. The general problem is to develop the concept of Property (T) and Ozawa's criterion for groupoids. Some conditions may be imposed on the groupoid: if it has a finite unit space, the extension should be straightforward. The next case would be a compact unit space equipped with a quasi-invariant measure. The application we have in mind is to a groupoid constructed from outer space, whose isotropy groups are isomorphic to $\mathrm{Out}(F_n)$. The unit space of the groupoid is the set of combinatorial types of graphs with fundamental group isomorphic to F_n; and a generating collection of morphisms in the groupoid are given by expansion/contraction of an edge. This is useful because all relations may be taken of length 3 in the generators, in contrast in particular to $\mathrm{Out}(F_4)$. This fits with an implicit requirement in any application of Ozawa's method: the sum-of-squares decomposition of $\Delta^2 - \epsilon\Delta$ only "sees" relations of short length in the support of Δ.

The symmetry groups of the combinatorial types of graphs appearing as unit spaces can also be exploited to decompose Δ into eigenspaces; this is a more involved Fourier decomposition, because morphisms have source and destination graphs, leading to two group actions on the space of morphisms.

The Cayley graph Laplacian above is just a very special case of the combinatorial Laplacian of the cellular L^2-chain complex of a Γ-covering of a finite CW-complex, and the combinatorial versions of the L^2-invariants are spectral invariants of these operators. For example, by a result of Varopoulos the 0-th Novikov–Shubin invariant α_0 (which describes the growth of the spectrum of the graph Laplacian near 0) is $+\infty$ except if the group is infinite and of polynomial growth, in which case α_0 is the polynomial growth rate.

Many of the driving structural questions about L^2-invariants translate to rather subtle questions about the spectrum of the Cayley graph Laplacian and of more general matrices over the integral group ring of the group in question. A negative answer to these questions typically has two aspects: first specific constructions of the groups (and the operator), then explicit spectral computations using adapted tools.

For example, we know that in general L^2-Betti numbers are not always rational, or even algebraic [TS9, 3, 41]. This answers negatively a famous question of Atiyah, which notwithstanding has a positive answer for many classes of torsion-free groups (compare, for instance, [TS6, TS7]). The corresponding question for the intriguing case of finite characteristic coefficients is adressed in [TS3]. All these results are inspired by work of Dicks and Schick [TS2], where the spectrum of the graph Laplacian on the lamplighter group is explicitly calculated, using mainly Fourier transform techniques for the base group to reduce to finite matrix calculations. Refining the constructions, Grabowski in [42] was even able to obtain examples with Novikov–Shubin invariant equal to 0, disproving a conjecture of Lott and Lück. In the converse direction, using ideas inspired by Voiculescu's R-transform and the theory of formal languages, Sauer proves in [95] that Novikov–Shubin invariants for free fundamental groups are always positive, and even rational. Previously, Lott had proved the same for abelian fundamental groups using Bloch–Floquet theory and perturbation theory for spectra.

The time is ripe to cover much more general cases. The main part of the project will be the development of the appropriate techniques. Our starting point here is that the surface groups are obtained as amalgamated products of two free groups, with amalgamation along an infinite cyclic subgroup, and the structure of the random walk operator is well adapted to this amalgamated free product decomposition. There is a version of Voiculescu's R-transform in such situations. It is a kind of free non-commutative Fourier transform, which has to be pushed to non-free situations, compare [110] for basics and [37] for first applications to spectral radius of amalgamated free products. In more detail, one looks at a group G, a subgroup H, and studies the compression to $\mathrm{End}(\mathbb{C}H)$ of an operator on G. The case of interest for us is G free and H a cyclic subgroup. A language has to be developed — based on context-free languages, D-finite languages, and so on — to encode the calculation of this compression, and to determine enough information about the spectrum of the initial combinatorial Laplacian. We hope to be able to extend the results of Sauer about rationality from free groups to surface groups (and beyond). Conversely, for general groups we now know that the positivity conjecture for Novikov–Shubin invariants is wrong, but we have no examples where the invariant (which is hard to compute explicitly) is known to be irrational. It should be possible to construct such examples based on the constructions and calculations in [42].

The project will profit from the help and expertise of associated researcher Laurent Bartholdi, who has significant expertise in spectral computations, using in particular also numerical methods.

We suggest for example the following thesis projects

- Property (T) for finitely generated groupoids and Ozawa's criterion.
- Outer Space of F_4, its groupoid, and Property (T).
- Voiculescu's R-transform for surface groups as amalgamated products and rationality of Green functions and Novikov–Shubin invariants for these groups.
- Irrationality of Novikov–Shubin invariants for lamplighter-like groups.

2.5 Representation theory for Lie algebroids (Jotz-Lean, Meyer, Zhu)

Let G be a simply connected Lie group and $\mathfrak{g}$ its Lie algebra. A continuous unitary representation of G on a Hilbert space $\mathcal{H}$ may be differentiated to a representation of $\mathfrak{g}$ or, equivalently, of the universal enveloping algebra $U(\mathfrak{g})$. This is a representation by unbounded operators defined on a common dense domain, the subspace of analytic vectors (see [97, Chapter 10]). Define Nelson's Laplacian Δ by $\Delta = \sum_{j=1}^{n} X_j^2$ for a basis $X_1, \dots, X_n$ of $\mathfrak{g}$. Nelson's Theorem says that a representation of $U(\mathfrak{g})$ comes from a representation of G if and only if Δ acts by an essentially self-adjoint operator. In this project, we propose to generalize Nelson's Theorem to *-algebras that appear in geometric quantisation.

As a first example, consider the *-algebra $\mathrm{Diff}(M)$ of differential operators on a manifold instead of $U(\mathfrak{g})$. The description of differential operators through symbols identifies $\mathrm{Diff}(M)$ with a certain vector space $S(T^*M)$ of functions on the cotangent space T^*M and gives a noncommutative *-algebra structure on $S(T^*M)$. A representation of $\mathrm{Diff}(M)$ maps functions in $S(T^*M)$ to operators on Hilbert space, thus providing a "quantisation map". Besides $\mathrm{Diff}(M)$, we also want a C*-algebra of observables that acts by bounded operators. The standard choice in case of T^*M is the C*-algebra $\mathbb{K}(L^2M)$ of compact operators on L^2M. Any representation of $\mathbb{K}(L^2M)$ is a direct sum of copies of the standard representation of $\mathbb{K}(L^2M)$ on L^2M. Such a representation clearly differentiates to a densely defined representation of $\mathrm{Diff}(M)$. More generally, a flat connection on a locally trivial Hilbert space bundle

$E \twoheadrightarrow M$ defines an action of $\mathrm{Diff}(M)$ on smooth sections of E. The groupoid C^*-algebra $\mathrm{C}^*(\Pi_1(M))$ of the fundamental groupoid $\Pi_1(M)$ of M acts naturally on the L^2-sections of such a bundle E and therefore seems a better C^*-algebra of observables than $\mathbb{K}(L^2M)$. The Lie groupoid $\Pi_1(M)$ is the unique one with simply connected source fibres and with TM as its Lie algebroid.

Let $\Delta \in \mathrm{Diff}(M)$ be the Laplace operator for some Riemannian metric on M. Call a representation of $\mathrm{Diff}(M)$ "integrable" if Δ acts by an essentially self-adjoint operator. We conjecture that these "integrable" representations of $\mathrm{Diff}(M)$ are equivalent to representations of $\mathrm{C}^*(\Pi_1(M))$. More precisely, a notion of C^*-hull for a class of "integrable" representations of a *-algebra is defined in [RM2]. We conjecture that $\mathrm{C}^*(\Pi_1(M))$ is a C^*-hull in this sense. The definition in [RM2] makes the C^*-hull unique and functorial, and it allows to prove a rather general induction theorem for C^*-hulls of algebras graded by a discrete group, which improves upon a result by Savchuk and Schmüdgen [96].

The algebras $U(\mathfrak{g})$ and $\mathrm{Diff}(M)$ above have a common generalisation. Namely, let G be a Lie groupoid with simply connected source fibres and let $A(G)$ be its Lie algebroid. Let $\mathrm{Diff}_G(G)$ be the algebra of left-invariant differential operators on G. This specializes to $U(\mathfrak{g})$ if G is a Lie group and to the algebra $\mathrm{Diff}(M)$ if $G = \Pi_1(M)$. There is an element $L \in \mathrm{Diff}_G(G)$ of order 2 that is elliptic along the range fibres of G. We conjecture that $\mathrm{C}^*(G)$ is a C^*-hull for the class of representations of $\mathrm{Diff}_G(G)$ in which L acts by an essentially self-adjoint operator. The most promising tools to establish this are the results of Woronowicz [118] about C^*-algebras generated by unbounded multipliers, combined with the pseudodifferential calculus for groupoids in [113]. The proof of a variant of Nelson's Theorem for representations on Hilbert modules by Pierrot [87] and the work of PI Meyer on C^*-algebras related to groupoids (see, for instance, [RM1, RM3]) are also relevant.

Which elements of $\mathrm{Diff}_G(G)$ may play the role of the Laplacian? For Lie algebra representations, this question has received much attention in mathematical physics. For instance, Nelson's Theorem for Lie algebra representations remains true if Nelson's Laplacian is replaced by the sum of squares over a set of Lie algebra generators of $\mathfrak{g}$ (see [107]). Since hypoellipticity is crucial in the proof of Nelson's Theorem, it should be studied whether any fibrewise hypoelliptic element of $\mathrm{Diff}_G(G)$ may be used to define integrability. This is particularly interesting for graded nilpotent groups, where hypoellipticity is equivalent to a representation-theoretic condition, the Rockland condition. The spectral theory of operators on such groups is also discussed in project area 2.6, and pseudodifferential calculi for such situations are important in project area 2.8.

The idea of quantisation is to build quantum mechanical observable algebras from a suitable structure on the phase space of a classical mechanical system. The algebras $\mathrm{Diff}(M)$ and $\mathbb{K}(L^2M)$ or $\mathrm{C}^*(\Pi_1(M))$ realize this goal if the phase space is T^*M with the canonical symplectic structure, and $\mathrm{Diff}_G(G)$ and $\mathrm{C}^*(G)$ above do so in slightly more complicated cases. Kontsevich showed that a Poisson structure suffices to build a formal deformation quantisation. But the convergence of his formal power series cannot be controlled. More structure seems needed to get an observable algebra for $\hbar \neq 0$. Weinstein suggested to quantise a Poisson manifold using a suitable symplectic groupoid G with a polarisation, that is, an involutive, multiplicative, Hermitian and Lagrangian distribution $\mathcal{P} \subseteq T_{\mathbb{C}}G$. While this approach covers many nice examples, several steps in the construction need technical extra assumptions to overcome analytic problems (see [50]).

We propose to split the geometric quantisation scheme in [50] into two steps. The first step builds a *-algebra $\mathcal{D}$ like $\mathrm{Diff}_G(G)$, and the second builds a C^*-hull for $\mathcal{D}$ like $\mathrm{C}^*(G)$. The construction of $\mathcal{D}$ is more geometric, and some of the analytic difficulties in Weinstein's geometric quantisation scheme in [50] are shifted to the problem of integrating $\mathcal{D}$ to a C^*-algebra, for which [RM2] provides useful tools.

The *-algebra $\mathrm{Diff}_G(G)$ for a Lie groupoid G is the enveloping *-algebra of the Lie algebroid of G. This definition still works for a Lie algebroid A that does not integrate to a Lie groupoid and so avoids the integrability obstruction in [50]. In addition, geometric quantisation uses the polarisation to define a kind of quotient of a Lie groupoid. For instance, if M is simply connected, the symplectic groupoid associated to T^*M is the pair groupoid $T^*M \times T^*M$, and the standard polarisation reduces this to the pair groupoid $M \times M$, which generates the C^*-algebra $\mathbb{K}(L^2M)$. The quotient groupoid above only exists if the leaf space of a certain foliation defined by the polarisation is a manifold (see also [JL1]). Real polarisations correspond to infinitesimal ideal systems on the Lie algebroid level (see [JL2, 50]).

Without regularity of the foliation, we may define $\mathcal{D}$ as the space of sections of the Lie algebroid that are in the kernel of a canonical flat connection defined in [JL2]. A construction along these lines should be tested for concrete non-integrable Poisson manifolds such as those in [CZ2, CZ3]. It is also interesting in this connection to study low-dimensional examples of Lie algebroids more systematically. The easiest case beyond Lie algebras are Lie algebroids with the circle as object space.

The Laplacian element in the universal enveloping algebra of a Lie algebroid and the resulting "integrable" representations may be defined without mentioning an integrating Lie groupoid. So any Lie algebroid defines a *-algebra and a class of "integrable" representations. What is a C^*-hull for these representations? A first guess is to build it from the "Weinstein groupoid" found in [CZ1] as an integrating object. For *-algebras defined by a Lie algebroid with an infinitesimal ideal system, it is less clear what to do. It seems best to first study this question in examples.

The induction theorem in [RM2, 96] works particularly well for a *-algebra with a $\mathbb{Z}^n$-grading and commutative degree-0 part. This is equivalent to an action of the torus $\mathbb{T}^n$ with commutative fixed-point algebra. In geometric examples with many symmetries, we expect a larger compact (quantum) group of symmetries. The proof of the induction theorem for C^*-hulls in [RM2] should extend to the setting of a *-subalgebra $A \subseteq B$ with a conditional expectation $B \to A$. This would cover, in particular, the case of compact quantum group actions. Such an induction theorem may also apply to the representation theory of the Drinfeld–Jimbo quantum groups $U_q(\mathfrak{g})$ because the fixed point algebra for the conjugation action of the quantum group on its enveloping algebra is the centre of $U_q(\mathfrak{g})$, hence commutative. The low-dimensional cases $\mathfrak{su}_q(2)$ and $\mathfrak{su}_q(1,1)$ are treated already in [96], using the conjugation action of the maximal torus.

Possible titles of thesis projects in this direction are:

- Nelson's Theorem for representations of Lie algebroids and Lie groupoids.
- Polarisations and reduced convolution algebras for infinitesimal ideal systems.
- Lie algebroids over low-dimensional manifolds.
- Induction theorems for C^*-hulls and applications.

2.6 L^2-invariants and harmonic analysis (Meyer, Schick)

The L^2-invariants of a space we plan to address in this project are the L^2-Betti numbers (introduced by Atiyah), the Novikov–Shubin invariants, and the L^2-torsion, a secondary invariant defined only if the L^2-Betti numbers vanish. These invariants have been used in geometric topology for many decades. To define and compute them, techniques from many other fields are used, notably operator algebras and functional analysis, differential geometry, ring theory, and group theory. At the same time, they have numerous applications back into these fields and to arithmetic geometry.

The L^2-invariants of a manifold X are defined initially as invariants of the Laplace operator $\tilde{\Delta}$ on the Hilbert space of square-integrable differential forms on the universal cover $\tilde{X}$ of X. A combinatorial variant of the definition using a CW-complex structure on X is used in project area 2.4. The group von Neumann algebra of $\pi_1(X)$ has a finite trace. This allows to define a regularized dimension for modules over it, which are usually infinite-dimensional as vector spaces. In particular, the kth L^2-Betti number of X is the regularized dimension of the kernel of the Laplace operator $\tilde{\Delta}$ on differential k-forms. The Novikov–Shubin invariant measures the growth rate of the spectrum of $\tilde{\Delta}$ near 0, using the traces of the spectral projectors $\chi_{[0,\lambda]}(\tilde{\Delta})$ for $\lambda \searrow 0$. The full spectrum of special such operators will be the focus of part of project 2.7.

The most interesting cases where L^2-invariants have been computed directly are (locally) symmetric spaces, where tools from harmonic analysis are applied to the differential form Laplacian $\tilde{\Delta}$. Approximation results compare these invariants to invariants in particular of arithmetic quotients. This way, computations of L^2-invariants can have arithmetic applications, or can profit from arithmetic knowledge. Somewhat surprisingly, the picture for the globally symmetric case is not yet complete and some intriguing questions remain open. One focus of this project is to extend the list of direct computations to further cases.

The Harish-Chandra Plancherel formula and the full knowledge of the Casimir operator is used by Olbrich [82] to compute the L^2-Betti numbers, Novikov–Shubin invariants and L^2-torsion of all compact locally symmetric spaces with a real, connected, semi-simple and *linear* underlying Lie group G, completing previous work of Borel, Lott, Hess–Schick [TS5]. The invariants depend only on the dimension, the volume, the Euler characteristic of the compact dual of the symmetric space, and the fundamental rank. Our focus here is on the Novikov–Shubin invariants. Let n be the dimension of the symmetric space G/K, where K is maximal compact in G. Let $m := \mathrm{rk}_{\mathbb{C}} G - \mathrm{rk}_{\mathbb{C}} K$ be the fundamental rank. If $m > 0$, then the Novikov–Shubin invariants are equal to m for degrees in $[(n-m)/2, (n+m)/2-1]$, and the Laplacian is invertible (so that the Novikov–Shubin invariant is $+\infty$) outside these degrees.

The result of Olbrich does not treat non-linear groups G. Such groups occur as infinite coverings of linear groups with infinite fundamental group. The most interesting case is probably $G = \widetilde{\mathrm{SL}(2,\mathbb{R})}$, which gives one of the 8 possible geometries of compact 3-manifolds in the Thurston classification. Another important group is $\widetilde{\mathrm{SU}(2,2)}$, the universal covering of the conformal group of 4-dimensional Minkowski space-time in relativistic quantum mechanics. More generally, we will study the universal cover of $\mathrm{SU}(p,q)$, generalizing both $\mathrm{SU}(2,2)$ and $\mathrm{SU}(1,1) = \mathrm{SL}(2,\mathbb{R})$.

The computation of the Novikov–Shubin invariants in these cases is important because it extends our very short list of spaces where these mysterious invariants are known. It is also a test case for a question of Gromov, namely, whether Novikov–Shubin invariants are invariant under quasi-isometries. This question is open mainly due to the lack of known values. Let $\Gamma \subseteq G$ be a cocompact subgroup and let $\widetilde{\Gamma} \subseteq \widetilde{G}$ be its inverse image in $\widetilde{G}$. This is an extension of Γ by a central infinite cyclic group. It turns out that $\widetilde{\Gamma}$ and $\Gamma \times \mathbb{Z}$ are quasi-isometric if G has property (T). In this way, the computation of the Novikov–Shubin invariants of $\widetilde{\Gamma}\backslash\widetilde{G}/K$, which are equal to those of the group $\widetilde{\Gamma}$, will be a test case for Gromov's question. We actually expect to find counterexamples. The values of the Novikov–Shubin invariants for $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ are stated in [69] (and compatible with a positive answer to Gromov's question) without any details of the computation.

A clear strategy to compute Novikov–Shubin invariants is explained already in [82]. The main problem is the explicit computation of the Plancherel measure. The classical work of Harish-Chandra, which is used in [82], only covers linear groups. There is a general Plancherel formula covering also the non-linear case, compare [34, 53]. The main work will be to make this explicit and usable for the calculation of the Novikov–Shubin invariants. Good test cases are $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ and $\widetilde{\mathrm{SU}(2,2)}$, for which the Plancherel measure is worked out explicitly in [52, 91]. Another ingredient for the calculations is $(\mathfrak{g}, K)$-cohomology, which is already well established. A previous thesis in Göttingen in a similar direction is that of Kammeyer [57], supervised by Schick and Meyer, in which Novikov–Shubin invariants and L^2-torsion have been computed for many non-uniform lattices in semi-simple Lie groups. This requires a precise understanding of the interplay of harmonic analysis with the Borel–Serre compactification. Young researchers in the groups of Bahns also use Plancherel measures in different contexts.

Another rich class of symmetric spaces where L^2-invariants give interesting information about the geometry are those nilpotent Lie groups which admit a cocompact lattice. This is, in a certain sense, at the opposite end from the semi-simple case discussed above, and very different techniques are needed. The Novikov–Shubin invariants of nilpotent Lie groups form "boundary contributions" for the study of Novikov–Shubin invariants of symmetric spaces of general semi-simple Lie groups in light of the boundary components of their Borel–Serre compactifications, compare Kammeyer's thesis [58].

By a classical result of Varopoulos, the 0th Novikov–Shubin invariant is the polynomial growth rate of the volume of balls or, equivalently, the rate of escape of the random walk on the group. Indeed, the 0th Novikov–Shubin invariant of a compact CW-complex is finite if and only if the fundamental group is infinite virtually nilpotent. Note that, by classical results, the polynomial growth rate of a discrete nilpotent group Γ is equal to an algebraic invariant given in terms of the nilpotency defining central series.

We know much less about the higher Novikov–Shubin invariants of nilpotent Lie groups (or equivalently their lattices). The best results about them have been obtained by Rumin. First, he computed explicitly all Novikov–Shubin invariants of the Heisenberg groups in [93], with considerable generalizations in [94]. In particular, [94] shows that for a graded nilpotent Lie group each of the Novikov–Shubin invariants is bounded above by the growth rate, and equality holds for the Laplacian on 1-forms on Lie groups with a quadratic presentation.

For the Heisenberg group, the Novikov–Shubin invariants are studied using harmonic analysis on the group, with partial information by Lott [68], and somewhat more completely – but with gaps in the proofs – in [103]. The more general results in [93, 94] still use some harmonic analysis. They depend on the canonical homogeneous structure on a graded nilpotent Lie group and the resulting filtration of the differential forms. The pseudodifferential calculus for homogeneous (graded) manifolds is also used to deal with auxiliary hypoelliptic operators, as well as homological algebra for Hilbert complexes.

In this project, we will refine the methods of Rumin for graded nilpotent groups, using the more refined calculi for those spaces available today, relying on the general theory developed, for instance, in [39]. These aspects also play a crucial role in Section 2.8. A graded structure is crucial in [94]. In general, of course, nilpotent Lie groups are only filtered. We will investigate how much the methods may be generalized to this case. A particularly interesting question is how the invariants for a filtered Lie algebra are related to those of the associated graded Lie algebra. To get finer information, we will combine the homogeneous structure and harmonic analysis.

The project area at hand investigates very fine spectral invariants. The harmonic analysis that enters this study may also be used to describe the K-homology class of an invariant, hypoelliptic differential operator on a graded nilpotent Lie group G_m. These differential operators are studied via their parametrices using adapted pseudodifferential calculi for filtered manifolds, apart from [39] one can also use the approach proposed in [38] using suitable groupoids. In an ongoing doctoral thesis project, these calculi are investigated by combining the groupoid approach with Rieffel's construction of generalised fixed point algebras. In the RTG, we want to push this further and use the techniques to solve index problems on filtered manifolds. The PIs have considerable expertise with this K-theoretic machinery and the groupoid approach to index theory (compare, for instance, [ES1, TS1, ES4, RM4]). However, the general techniques alone cannot solve the problem because their end result still involves a map that is only defined as the inverse of a certain isomorphism. Namely, the principal symbol belongs to a certain non-commutative C^*-algebra. Its K-theory is isomorphic to that of the unit cosphere bundle S^*M in the manifold M. But which K-theory class on S^*M corresponds to a given principal symbol? The Rockland condition, which is the analogue of ellipticity in this context, asks for the symbol to be invertible. The problem is to use the Rockland condition to describe the class through finite-dimensional data extracted from the operator, in a way that still works for bundles of graded nilpotent Lie groups. This is a key ingredient in the recent hypoelliptic index theorem by Baum and van Erp [7].

The development of the above theory offers many attractive questions for several doctoral students. Preliminary titles of thesis projects in this direction are:

- Plancherel measure and L^2-invariants of non-linear semi-simple Lie groups.
- Novikov–Shubin invariants of nilpotent Lie groups via filtered calculi and harmonic analysis.
- Index theory for hypoelliptic invariant differential operators in graded nilpotent Lie group bundles.

2.7 Spectral engineering (Schick, Schrohe, Witt)

Originally motivated by solid state physics, in this project we are interested in spectral properties of geometric differential operators invariant under a cocompact discrete group action. The original example is the Laplacian on Euclidean space with a $\mathbb{Z}^n$-invariant potential. And the question is: can we (at least in a certain spectral range) achieve a determined band-gap structure of the spectrum of this operator.

More generally, a very similar type of operators occurs when introducing and studying analytic L^2-invariants (in the sense of Atiyah) (see 2.4 and 2.6). The starting point is a a normal covering $\bar{M} \to M$ of a compact Riemannian manifold M with an action of the deck-transformation group Γ. The relevant operator now is the differential form Laplacian on $\bar{M}$ or, more generally, the lift $\bar{D}$ of an elliptic differential operator D on M. Classical L^2-invariants depend on the spectrum near zero. We would like to understand more about the full spectrum of $\bar{D}$. In particular, to what extent can we arrange for a lower bound on the number of gaps (within a spectral range)? For this spectral engineering problem, the role of the potential of the classical problem is played by the metric (and, in addition, also the topology of M): can we choose the metric so as to achieve a predetermined band-gap structure of the spectrum of $\bar{D}$? Or are there obstructions, forcing the spectrum, say, to be the full (half)-line?

This question has a considerable history. When the group Γ is $\mathbb{Z}^n$ with $n \neq 0$, Post in [89] solves the problem positively for the scalar Laplacian. Using Fourier analysis in the form of Bloch–Floquet theory, for a given finite energy range Λ, he constructs (M, g) with $\mathbb{Z}^n$-covering $\bar{M}$ and such that the spectrum of the scalar Laplacian on $\bar{M}$ has a prescribed number (and approximate location) of gaps in the interval $[0, \Lambda]$. For a very specific type of manifold, this is refined by Khrabustovskyi, who completely prescribes the band-structure of the spectrum in any finite energy range [61]. Again, this relies heavily on Fourier analysis. The method of Post, however, is more flexible. The paper [90] shows, in particular, that one may prescribe the manifold. Then a suitable conformal change of the metric achieves the desired band-gap structure. Using a non-commutative version of Bloch–Floquet theory for a group Γ which is a finite extensions of $\mathbb{Z}^n$, [66] generalizes the results of [89] to such Γ as symmetry group, and further in [67] to residually finite symmetry groups. A final result is obtained in recent work by Schoen and Tran [98] who construct, for an arbitrary covering $\bar{M} \to M$ of a compact manifold and an arbitrary L, a metric on M such that the scalar Laplacian for the lifted metric on $\bar{M}$ has at least L gaps in its L^2-essential spectrum. The main point is that there is no condition whatsoever on the covering group.

The scalar Laplacian is only the first in the list of important geometric differential operators. The differential form Laplace–Beltrami operators and the spin Dirac operator of a spin structure offer the next generation of examples. The spectrum of these basic geometric operators should depend strongly on the metric. Only very little is known, however, about spectral engineering for these operators. Recently, Egidi and Post produced metrics on compact manifolds with large gaps in the spectrum of the Hodge Laplacian (a weak analogue for differential forms of a celebrated result of Colin de Verdière), but only on quite special types of manifolds. Metrics with an arbitrary number of gaps in the spectrum of differential form Laplacians and Dirac operators are constructed in [2], but only for $\mathbb{Z}$-symmetry. The construction requires certain topological conditions on a separating hypersurface. It should also be noted here that index theory gives topological obstructions to the existence of gaps in the spectrum of the Dirac operator. Schick has significantly contributed to the identification of such obstructions via index theory (for example, in [TS4, TS8]) with a particular emphasis on spectral methods and Fourier decomposition. Therefore, the constructions for general operators need to be more sophisticated than for the scalar Laplacian, where the previous work shows that no such obstructions exist.

We have considerable experience in index theory and spectral theory of non-compact manifolds and general operators. Based on this, thesis projects, supervised by Schick, Schrohe, and Witt, will concern the study of Dirac and differential form Laplacians with more general symmetry group Γ, to identify, on the one hand, obstructions to band-gap structure and, on the other hand, construct examples with many gaps in the spectrum when the obstructions vanish (spectral engineering). The precise results of [61] rely on the full power of Bloch–Floquet theory. In a second line of projects, we will refine these techniques in two directions: to more general symmetry groups Γ (for instance, virtually nilpotent groups as studied in 2.6) on the one hand, to more general operators (differential form Laplacian, Dirac operator) on the other hand, and construct metrics with prescribed band-gap structure of these operators on Γ-coverings. In all cases, the construction part will involve a family of metrics which degenerates in certain parts of the manifold and such that the spectrum of the operator in question (differential form Laplacian, Dirac operator, ...) converges to the spectrum of a model operator which can be computed explicitly. For abelian groups, Fourier analysis allows to carry out these delicate computations on compact manifolds, which simplifies the situation and therefore will be the first case to be studied. The presence of obstructions to the existence of gaps is somewhat hard to pin down and will force us to start with special cases, like the n-torus, where we expect that specific constructions like Khrabustovskyi's will allow to control the spectrum of the differential form Laplacians.

Discretisation is a complementary approach to the analysis of spectral properties of Laplacians (and more general operators). It is quite subtle to find discretisation techniques that give good approximation results for large parts of the spectrum of the differential operator by the discrete analogues. Note that, typically, the discrete operators will be bounded, so that we cannot expect to approximate the full spectrum at once. For the zero eigenvalues, the Hodge–de Rham Theorem provides a perfect discretisation method: the combinatorial Laplacian of any triangulation has the same kernel as the differential form Laplacian of the appropriate degree. This holds for compact manifolds, but equally so for the symmetric Laplacians on coverings discussed so far, due to Dodziuk's L^2-Hodge–de Rham Theorem [31].

However, Dodziuk and Patodi [32, 33] obtained a much more refined spectral approximation result for compact manifolds: given any compact Riemannian manifold and finer and finer triangulations which are sufficiently regular, then for each k, the kth eigenvalue of the combinatorial Laplacian converges to the kth eigenvalue of the Hodge Laplacian, and this with precise error bounds. In particular, the convergence is uniform on any finite part of the spectrum. This spectral computation uses Rayleigh quotient computations and the precise analysis of the de Rham map and its explicit homotopy inverse constructed by Whitney.

Obviously, it does not make sense to aim for a similarly formulated spectral approximation result for the operators on coverings, as they have continuous spectrum in general. A substitute is the spectral density function and, for $\mathbb{Z}^n$-symmetry, the individual terms in the Bloch–Floquet decomposition. Still, one has to formulate the spectral convergence statement carefully. The Rayleigh quotient considerations of the compact case are appropriate for eigenvalue estimates, but again have to be replaced by a more functional analytic treatment for operators with continuous spectrum. We are optimistic that these difficulties can be overcome and propose this as a further thesis topic. One also has to develop the appropriate discrete version of the twisting with a flat representation, which is another subject interesting in its own right.

List of potential thesis topics:

- Fine spectral engineering for $\mathbb{Z}^n$-invariant differential form Laplacians using Bloch–Floquet theory, in particular, on $\mathbb{R}^n$.
- Spectral engineering for form Laplacians: constructions for arbitrary covering spaces.
- Spectral engineering for Dirac operators: index obstructions versus constructions for abelian and non-abelian coverings.
- Riemannian structures and triangulations of manifolds for covering spaces.

2.8 Resolvent and dispersive estimates (Schrohe, Witt)

This project aims at studying the resolvent structure for certain singular geometries, and also to address dispersive properties of operators in the corresponding pseudodifferential calculi using tools from Fourier analysis. The emphasis is on the use of oscillatory integrals, spectral theory, and Fourier techniques, all at the core of the proposed RTG.

An approach to understand the spectral and scattering theory of a geometrically interesting (positive) elliptic, or hypoelliptic, differential operator A, like the Laplacian, or the sub-Laplacian, is to understand the resolvent $R(\lambda) = (A - \lambda)^{-1}$. In many situations, a first step is to regard this resolvent $R(\lambda)$ as a parameter-dependent family of pseudodifferential operators, which then opens up a whole arsenal of microlocal techniques. A first goal is achieved once one has succeeded in constructing a parametrix to $R(\lambda)$ in a symbolic manner. For the case of manifolds with conical singularities, this analysis has been carried out in [ES7], [102]; see also [ES5] for applications.

A parametrix construction is particularly intricate on non-compact manifolds, where the geometry at infinity plays a decisive role. Notice that the spectrum of A then has an absolutely continuous part $\sigma_{\mathsf{ac}}(A)$, while there may also be discrete spectrum (below the essential spectrum or embedded into it). Whereas number theory is mostly interested in the discrete spectrum, scattering theory concerns the absolutely continuous part of the spectrum (according to Melrose [76], scattering theory provides a parametrisation of the absolutely continuous spectrum).

It has been proven natural to consider classes of Riemannian manifolds with an asymptotic control of the metric at infinity. In this project, we will focus on the following three instances: (1) asymptotically Euclidean manifolds, for which one has the SG (or scattering) calculus [75, 99], (2) asymptotically hyperbolic manifolds, for which the zero calculus of Mazzeo and Melrose [14, 43, 74] has been developed, and (3) homogeneous nilpotent Lie groups as discussed also in Sections 2.5 and 2.6, following recent work [39, 88] by Fischer, Ruzhansky, and others. The first two calculi already exist in a semiclassical form [114, 116]. For instance, in the first case one would consider $(-h^2\Delta - 1)^{-1}$ instead of $(-\Delta - \lambda)^{-1}$, where $h = 1/\sqrt{\lambda} > 0$ plays the role of the semiclassical parameter. Note that (the symbol $|\xi|^2 - 1$ of) the semiclassical operator $-h^2\Delta - 1$ is non-elliptic, but of real-principal type. This means that microlocally the non-elliptic points are still "nice" in the sense that there are methods available to readily obtain all the required information. Such situations have been handled by various authors through elliptic estimates, results on the propagation of singularities, and complex scaling (see [108] for the latter). Recently, Vasy [114, 121] has devised a method that provides a systematic framework for arguments along these lines. One particular point here is that extending the operators under consideration beyond a natural boundary makes it necessary to study the propagation of singularities near radial points. This has been done so far using positive commutator estimates [47]. It may also be achievable with a parametrix construction [IW3].

In this project, our approach will be a different one. Following a general outline as given, for instance, in [100, 101, 105], we will keep the form $(A - \lambda)^{-1}$ of the resolvent, but consider it in conical regions $\Lambda \subset \mathbb{C}$ for the spectral parameter λ, where the operator $A - \lambda$ is parameter-dependent elliptic. This will allow us to reach similar conclusions as in the papers [114, 121] mentioned above, but also to go further. Our approach has the advantage of a greater flexibility in identifying symbolic components, which as a consequence allows us to exercise some extra control on the problems under investigation.

In recent years, it has been realized in different places [28, 36, 49, 119] that dynamical properties of the characteristic flow play an important role if one wants to obtain the most refined resolvent estimates. A famous example is quantum ergodicity [1, 81], which holds on a closed manifold if the characteristic flow is ergodic and where, on average, eigenfunctions become equidistributed in phase space, in the high-energy limit. We will pay special attention to such dynamical properties.

We will likewise investigate time-dependent operators like $\partial_t^2 + A$ or $\mathrm{i}\partial_t + A$, where A is as above. Here the goal is to prove new dispersive estimates on the solutions of the corresponding evolution equations (often called waves) or to improve existing ones (for instance, concerning the parameter range, where these dispersive estimates are known to be valid). A key example are Strichartz estimates (see, for instance, [5, 48]), which assert that certain space-time averages of the solutions behave better (in terms of decay) than one would expect from just concentrating on the solutions at fixed times. Strichartz estimates for certain degenerate hyperbolic operators were proven in [IW1, IW2]. In the examples above, conservation of energy holds. This is why one sees no dispersive effects by solely employing L^2-based norms with respect to the spatial variables. The situation, however, starts to improve if one replaces the L^2 norm by the L^∞ norm, where one already sees the pointwise decay of the solutions. This observation will be the point of departure for a whole series of refinements.

Here our approach will take advantage of the Lagrangian structure [6, 109] of the (distributional) kernels of the solving operators of the problems under study. This again involves symbolic aspects which become apparent by mentioning the appearance of eikonal and transport equations. Also dynamical aspects are present, e. g., as seen by the fact that in favorable cases the underlying Lagrange manifolds are embedded (instead of immersed) and globally well-behaved. The final step is to utilize the symbolic information gathered till then to derive the desired dispersive estimates. This step will heavily rely on tools from harmonic analysis [111].

Topics for prospective thesis projects include:

- Parameter-dependent pseudodifferential calculi as extensions of the calculi without a parameter and a symbolic parametrix construction.

- Resolvent estimates of the differential operators under study while making suitable dynamical assumptions.

- Dispersive estimates relying on the Lagrangian structure of the kernels of the operators under consideration.
- Spectral-theoretic consequences of the resolvent and dispersive estimates, like wave-trace invariants or the distribution of resonances.

A Publications and Bibliography

A.1 List of published previous research relevant to the research programme

a) Articles which at the time of proposal submission have been published or officially accepted by publication outlets with scientific quality assurance, and book publications.

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